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Small scale yielding conditions for steep residual stress distribution

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ABSTRACT

Small scale yielding conditions are generally used as a fracture mechanics parameter to assess the applicability of stress intensity factors. When a steep residual stress distribution exists near a crack tip, there is potential to deviate from the conditions, even for those that are satisfied by general elastic analysis. In this study, the features of the residual stress distribution shape which influence the scale of yielding are investigated by conducting numerical analysis considering elastoplastic behavior. Additionally, a simplified practical method to evaluate the condition is proposed when a steep residual stress distribution is present.

1. Introduction

The stress intensity factor (SIF) is a fracture mechanics parameter that describes the stress field around a crack tip. The SIF for a crack tip in an infinite plate under uniform single-axis tensile loading is obtained from the stress function proposed by Westergaard [1]. Irwin analyzed the plastic zone near a crack tip in an elastic-perfectly plastic material under small-scale yielding conditions [2]. The SIF is used as a parameter to evaluate time-independent and time-dependent fracture such as fatigue cracks or stress-corrosion cracking.

When the SIF is used as a fracture mechanics parameter, the small-scale yielding (SSY) condition in which the plastic zone size near the crack tip, should be sufficiently small relative to the crack length, must be satisfied. The plane strain fracture toughness testing in ASTM E399 [3], for instance, has a procedure to confirm the SSY validity after an experiment. When evaluating the crack growth rate for time-dependent fractures, the SSY conditions should also be satisfied.

When the plastic zone size near the crack tip becomes large enough, the condition is called large-scale yielding (LSY). These two conditions, SSY and LSY, however, cannot be separated clearly. When the SSY criterion, which is developed as an engineering rule such as in the ASTM standards, is exceeded, the scale of yielding is changed to the transition process from SSY to LSY. In these cases, the SIF is inadequate as a parameter for the assessment of crack propagation because the error from evaluation using the SIF gradually increases. In particular, when a steep residual stress distribution exists along a crack propagation path, the potential exists to partially exceed the SSY criterion at the region of high stress. Generally, even if the potential exists to fully exceed the SSY criterion in such cases, crack propagation assessment is conducted using the SIF alone. For instance, for stress-corrosion cracking of austenitic stainless steel with high-temperature water in nuclear power plants, large variations in residual stress along the crack propagation path are expected, nevertheless the SIF is generally used for evaluating the crack growth rate [4]. In such cases, the possibility that the evaluation using the SIF becomes inadequate should be considered. Parker [5] reported that the SIF can be written as the superposition of the SIF from remote uniform

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Nomenclature

a	half length of an actual crack (m)
A	coefficient of magnitude of the normal distribution
b_k	dimensions for a partially loaded crack (k = 1, 2) (m)
E	Young's modulus of elasticity (Pa)
F_j	boundary-correction factor on the stress intensity for element j
K	stress intensity factor (Pa m$^{1/2}$)
K_j	stress intensity factor on a segment of the crack surface for element j (Pa m$^{1/2}$)
$K(a)$	stress intensity factor at a (Pa m$^{1/2}$)
K_0	stress intensity factor derived from the remote uniform tensile stress (Pa m$^{1/2}$)
K_{res}	stress intensity factor derived from the residual tensile stress (Pa m$^{1/2}$)
M	coefficient of maximum value of the normal distribution
n	total number of bar elements
r	dimensions from an actual crack tip, $r = x - a$ (m)
S	standard deviation of the normal distribution
x	coordinate location (m)
W	specimen width (m)
δ	crack tip opening displacement (CTOD) (m)
Δ	difference between the peak position of the CTOD and μ
λ	plastic constraint factor
μ	mean value of the normal distribution (m)
σ_0	remote uniform tensile stress (Pa)
σ_j	stress on the segment of the crack surface for element j (Pa)
σ_{res}	residual tensile stress (Pa)
$\sigma_{res,co}$	residual compressive stress (Pa)
σ_Y	yield stress (Pa)
ω	plastic zone size (m)

stress and residual stress. However, the quantitative SSY criterion in the presence of residual stress distribution has not been clarified. Therefore, it is important to clarify the effect of the residual stress distribution on the SSY conditions and to establish the SSY criterion for which the residual stress distribution is present.

When the shape of the residual stress distribution is gradual, as presented in Fig. 1a, the conditions keep the SSY, and the plastic zone size can be obtained using elastic analysis. However, when the shape of the residual stress distribution is steep, as presented in Fig. 1b, the stress varies drastically near the crack tip. It affects the plastic zone size which should be determined by the elastoplastic analysis. The increases in the plastic zone size depend on the shape of the stress distribution. This behavior means a transition process for yielding from SSY to LSY. In other words, the error from the theoretical linear fracture mechanics increases and SIF is not appropriate for crack growth evaluation in some cases. Typical stress distribution parameters for the steep residual stress distribution that reach the SSY criterion may exist. In this study, various shapes

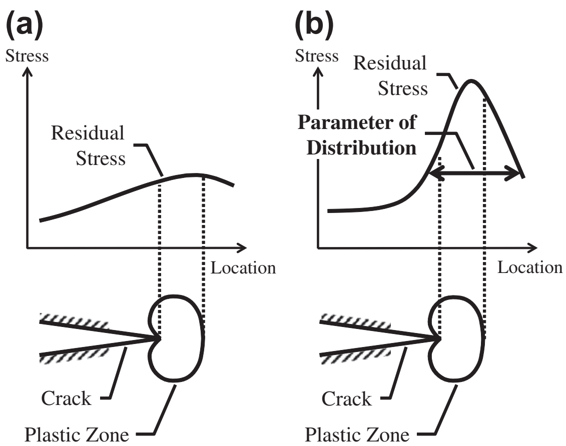

Fig. 1. Schematic diagram showing the positional relationship between the plastic zone and the residual stress distribution for: (a) the gentle distribution of the residual stress and (b) the steep distribution of the residual stress.

of residual stress distributions along the crack propagation path are considered in the crack propagation model based on the Dugdale model [6]. The influence of the distribution on the SSY conditions was evaluated analytically. In addition, the SSY criterion in the presence of the residual stress distribution was proposed and the effectiveness of the criterion was discussed.

2. Analytical model and procedure

Fig. 2 shows the state for a finite plate that has a center crack. The plate has been loaded with a remote uniform stress distribution and a residual stress distribution. A plastic zone is present in the vicinity of the crack tip.

Newman's model [7] was adopted as an analytical model because it can readily deal with stress distributed along the crack propagation path. Newman's model is a two-dimensional model based on the Dugdale model. In this model, the crack is divided into discrete bar elements. Each bar element has stress and a displacement. Because of the symmetry of the model, the analysis was performed for half of the plate ($W/2$). The plate was divided into n elements within the $W/2$; the remote stress σ_0 and the residual stress σ_{res} were divided discretely for each element. The stress value for each element was determined using the mean value of the stress at both ends of the element coordinates.

Fig. 3 is a schematic diagram of Newman's model applied to Fig. 2. The analysis was conducted to determine the mode I SIF value when the elements were broken up to the ith element ($i = 1$–n) starting from the center. The crack closure behavior is regarded as beyond the scope of this study. The constant stress σ_j for each bar element is defined as:

$$\sigma_j = \sigma_{0,j} + \sigma_{res,j}, \tag{1}$$

where $\sigma_{0,j}$ and $\sigma_{res,j}$ denote the discrete remote and discrete residual stresses. K_j, which is the SIF derived from σ_j, is given by Eqs. (2)–(5), where $K_{\infty,j}$ is the SIF derived from σ_j in the case of an infinite plate:

$$K_j = K_{\infty,j} F_j, \tag{2}$$

$$K_{\infty,j} = \frac{2\sigma_j}{\pi}\sqrt{\pi a}\left[\sin^{-1}\left(\frac{b_2}{a}\right) - \sin^{-1}\left(\frac{b_1}{a}\right)\right], \tag{3}$$

$$F_j = \left[\frac{\sin^{-1} B_2 - \sin^{-1} B_1}{\sin^{-1}\left(\frac{b_2}{a}\right) - \sin^{-1}\left(\frac{b_1}{a}\right)}\right]\sqrt{\sec\frac{\pi a}{W}}, \tag{4}$$

$$B_k = \frac{\sin\frac{\pi b_k}{W}}{\sin\frac{\pi a}{W}} \quad (k = 1, 2) \tag{5}$$

Finally, $K(a)$, which is the SIF value for the crack presented in Fig. 2, was calculated using the sum of the SIFs on the segment of the crack.

$$K(a) = \sum_{n=1}^{i} K_j. \tag{6}$$

The plastic zone size ω was then calculated. The elastic-perfectly plastic solid was considered. In the Dugdale model, the compressive yield stress was loaded at the notional crack surface, and the SIF became zero at the point $x = a + \omega$. $K_{\sigma Y}$,

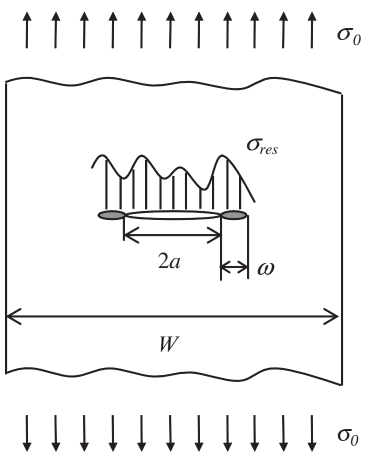

Fig. 2. Center crack tension specimen with residual stress and plastic zone.

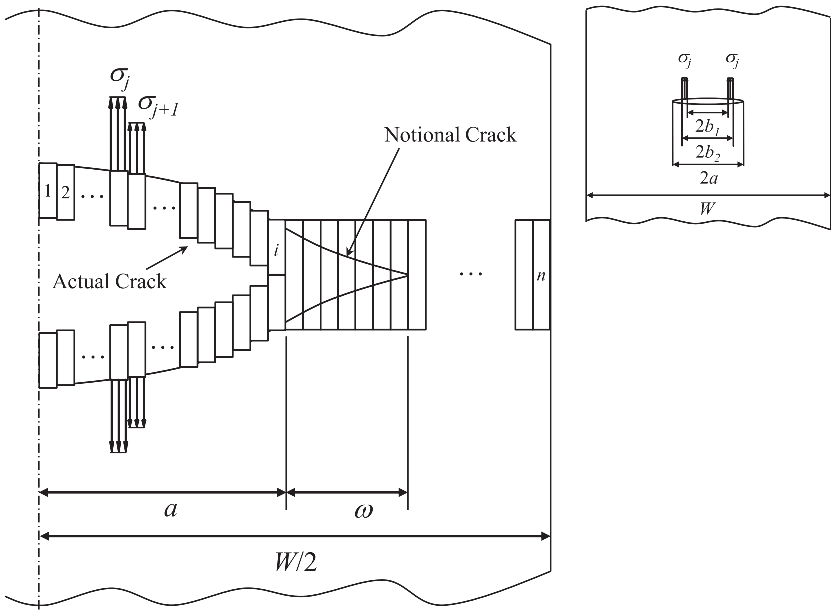

Fig. 3. Schematic diagram of Newman's model.

which is the SIF derived from the compressive yield stress loaded at the notional crack surface, obtained from Eqs. (2)–(5) as follows:

$$K_{\sigma Y} = -\lambda\sigma_Y\left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\sin\frac{\pi a}{W}}{\sin\frac{\pi(a+\omega)}{W}}\right)\right] \times \sqrt{\pi(a+\omega)\sec\frac{\pi(a+\omega)}{W}}. \tag{7}$$

The plastic zone size ω is obtained by solving the following equation using numerical calculations.

$$K(a+\omega) + K_{\sigma Y} = 0. \tag{8}$$

In addition, the CTOD is expressed for this model as [8]:

$$\delta = \frac{8\lambda\sigma_Y a}{\pi E}\ln\left(1 + \frac{\omega}{a}\right). \tag{9}$$

We conducted these calculations numerically using a C++ program. The material, stress and other properties used in the calculations are presented in Table 1.

3. Analytical results and discussion

Fig. 4 presents an example of analytical results obtained using Newman's model. The applied stress distribution is shown in the upper part of Fig. 4. It consists of the residual stress and the uniform remote stress. The residual stress distribution is represented as a normal distribution, written as:

$$\sigma_{res} = M\sigma_Y\exp\left(-\frac{(2a/W-\mu)^2}{2S^2}\right), \tag{10}$$

Table 1
Properties for analysis using the Newman model.

Typical material	304 SS
Yield stress, σ_Y	205 MPa
Young's modulus, E	193 GPa
Remote stress, σ_0	$\sigma_Y/4 - \sigma_{res,co}$
Residual stress, σ_{res}	Normal distribution
Plastic constraint factor, λ	$\sqrt{}(2\sqrt{}2)$
Specimen width, W	2
Number of elements, n	100,000

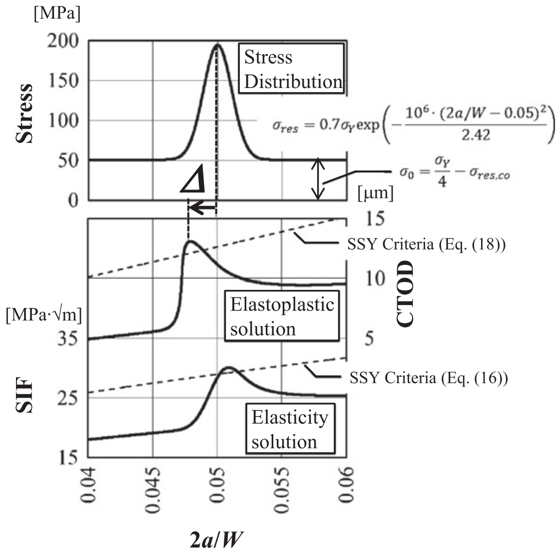

Fig. 4. Example of analytical results for: upper, stress distribution (=$\sigma_0 + \sigma_{res}$); middle, analytical results of CTOD as the elastoplastic solution; lower, analytical results of SIF as the elasticity solution.

$$M = \frac{1}{AS\sqrt{2\pi}}. \tag{11}$$

Here, $M\sigma Y$ signifies a maximum stress value of the residual stress distribution. In addition, the residual compressive stress is uniformly loaded for balancing the residual tensile stress. The residual compressive stress is written as:

$$\sigma_{res,co} = \frac{2\sigma_Y}{AW} \tag{12}$$

Fig. 4 portrays an example with parameters $M = 0.7$, $S = 0.0011$, and $\mu = 0.05$ as the residual stress distribution. The SIF and the CTOD were calculated when the actual crack had progressed to $2a/W$. The SIF results are shown at the bottom of Fig. 4 as the elasticity solution. The CTOD results are shown in the middle of Fig. 4 as the elastoplastic solution. Both the SIF and the CTOD steeply changed around the residual stress distribution. A local maximum value of the CTOD is located in the forward position of the stress distribution peak, and a local maximum value of the SIF is found at the back.

Now the SSY criterion for these results is considered. A definition of a small-scale yielding is:

$$\omega \ll a. \tag{13}$$

In the Irwin model, the following equation has been widely used for the SSY criterion as an engineering rule:

$$r \leqslant 0.02a. \tag{14}$$

This engenders the following equation [9]:

$$a \geqslant 2.5\left(\frac{K(a)}{\sigma_Y}\right)^2. \tag{15}$$

Eq. (15) can be transformed by:

$$K(a) \leqslant \sigma_Y\sqrt{0.4a}. \tag{16}$$

Eq. (16) shows the SSY criterion expressed using $K(a)$.

On the other hand, the SSY criterion expressed using the CTOD is derived from Eqs. (9) and (13) [8]:

$$\delta = \frac{8\lambda\sigma_Y\omega}{\pi E} = \frac{K(a)^2}{E\lambda\sigma_Y}. \tag{17}$$

By substituting Eqs. (17) into (15), the SSY criterion is obtained as shown below:

$$\delta \leqslant 0.4a\frac{\sigma_Y}{E\lambda}. \tag{18}$$

The calculated results of Eqs. (16) and (18) with the properties of Table 1 are shown by dashed lines in Fig. 4. The results indicate that the SIF or the CTOD increases around the residual stress distribution, and suggest that they exceed the SSY criterion locally. In Fig. 4, the difference between the local maximum locations of the SIF and CTOD curves is observed, although it is well known that the square of the SIF is proportional to the CTOD when the SSY condition is satisfied in general. This means the proportional relationship of the SIF and CTOD has deviated in Fig. 4 and the conditions have migrated to the transition conditions of LSY from SSY.

In the region that exceeds the SSY criterion locally, crack growth evaluation using the SIF is not adequate. The values of the stress distribution parameters that are determined by the SSY criterion may exist. The values of the distribution parameters which reached the SSY criterion were analyzed. Fig. 5 portrays the distribution parameters determined by the SSY criterion of Eq. (16), as defined by the standard deviation S and the center value of the residual stress distribution μ as a function of the maximum value of the residual stress distribution M. When the criterion is exceeded, the condition becomes the transition process of yielding from SSY to LSY. In these regions, the error from the theoretical analysis of the linear fracture mechanics increases.

As shown in Fig. 5, particularly when addressing the stress distribution standard deviation S, when μ is small, a smaller standard deviation S is necessary to satisfy the SSY conditions: satisfying the SSY becomes extremely difficult even if the residual stress distribution is gentle in the case of small μ. This is because the crack is too short for comparison with the plastic zone size because the crack propagation has just started. When μ becomes larger, the SSY condition is satisfied easily because the crack length is sufficiently large. However, with further increase of μ, the plastic zone size increases significantly at the location of the residual stress distribution. Therefore, satisfying the SSY becomes difficult again. However, particularly when addressing parameter M, the SSY is not satisfied with increasing M.

From these results, a structure with a residual stress distribution for which the gradient is steep or for which the maximum value is large tends to exceed the SSY criterion at the location of residual tensile stress. Especially when a large residual stress distribution is loaded near the position of crack generation, the possibility of exceeding the SSY criterion must be considered.

As shown in Fig. 4, the local maximum value of the CTOD, as the elastoplastic solution, shifted forward, which suggests that the position where the crack growth rate increases also shifted forward, and might imply a misleading rate when the SIF, as the elasticity solution, is used as the evaluation parameter. Therefore, Δ is defined as the difference between the peak position of the CTOD and μ, and the qualitative trends of Δ were examined. The stress distribution for the test is depicted in Fig. 6. Distribution I ($M = 0.40$) and II ($M = 0.08$) were compared for evaluating the effects of parameter M. Distribution II ($S = 0.001$) and III ($S = 0.005$) were compared to evaluate the effect of parameter S.

The results are portrayed in Fig. 7. When μ became larger, Δ also became larger for all the results because of the increasing plastic zone size at the location of the residual stress distribution. It is noteworthy that distribution I had the largest Δ of the three curves, and that distribution II had the second largest. Therefore, when the maximum value of residual stress was high or the standard deviation of the distribution was low, Δ became large. In other words, the CTOD as the elastoplastic solution increased before the crack tip reached the point where the residual stress distribution was present. The CTOD is assumed to increase the crack growth rate before the crack tip reaches such a position.

In general, the crack growth rate is calculated using the SIF for the remaining life evaluation of the structures. Since the SIF, as the elasticity solution, is not increased until the crack tip reaches the residual stress distribution, a risk of underestimating the remaining life of the structure exists when a residual stress distribution with a steep gradient or large maximum value is present.

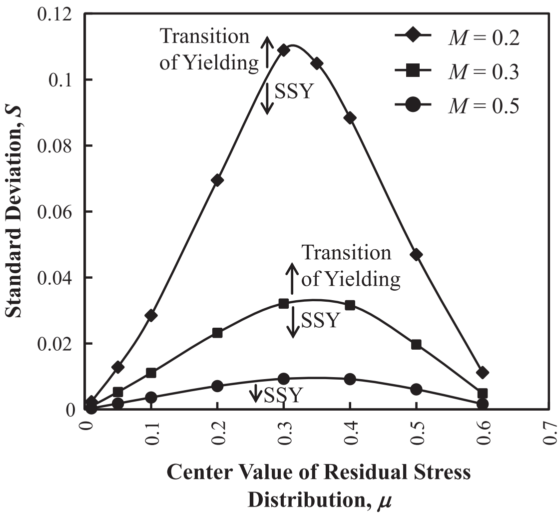

Fig. 5. Analytical results of the SSY criterion for different parameters of the normal distribution μ, S, and M.

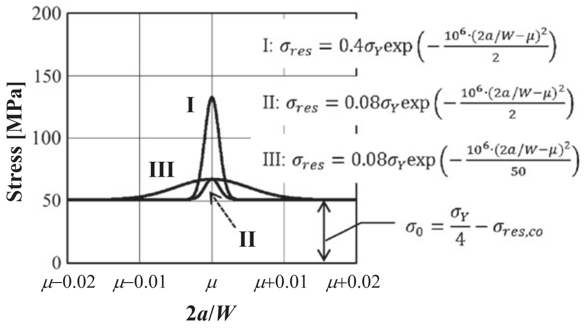

Fig. 6. Stress distribution for the qualitative assessment of Δ.

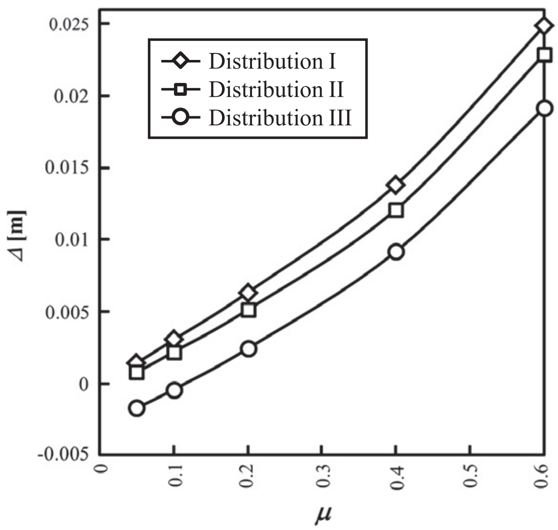

Fig. 7. Analytical results of Δ for the stress distribution depicted in Fig. 6.

Further discussion can be undertaken in terms of theoretical considerations. The plastic zone size is determined using Eq. (8) in the Dugdale model. In other words, the plastic zone size is determined at the intersection of the function $K_{\sigma Y}$ and the function $K(a + \omega)$.

A relevant schematic diagram is presented in Fig. 8. When the SIF varies gently at the plastic zone, as presented in Fig. 8a, the difference between $K(a_1)$ as the elasticity solution, which is the SIF at the actual crack length a_1, and $K(a_1 + \omega_1)$ as the elastoplastic solution, which is the SIF considering that the plastic zone size ω_1, is small. Therefore, the difference between

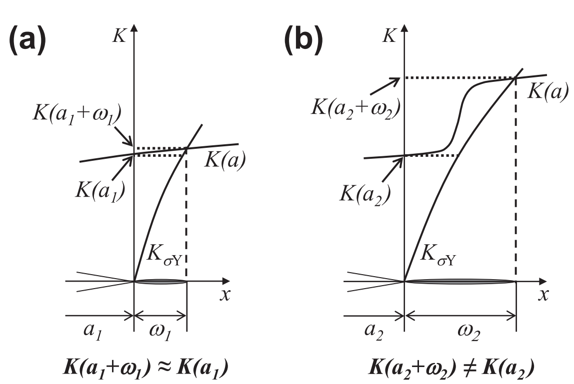

Fig. 8. Schematic diagram showing the relationship between the SIF function and plastic zone size in the Dugdale model for: (a) a low-slope SIF and (b) a steep-slope SIF.

the plastic zone size derived from $K(a_1)$ and $K(a_1+\omega_1)$ is also small. Consequently, $K(a_1)$ can be used as a fracture mechanics parameter. This condition indicates SSY.

However, when a steep residual stress distribution is loaded locally, the SIF varies considerably, as presented in Fig. 8b. In this case, the difference between $K(a_2)$ as the elasticity solution and $K(a_2+\omega_2)$ as the elastoplastic solution is large; and the plastic zone size cannot be calculated from $K(a_2)$. Consequently, $K(a_2)$ cannot be used as a fracture mechanics parameter. The elastoplastic behavior has been regarded as SIF-variant at the plastic zone. Therefore, the local maximum value of CTOD, which is the elastoplastic solution, will be shifted forward.

On these considerations, the traditional SSY criterion based on elastic analysis is inapplicable in the presence of a steep residual stress distribution. Therefore, the SSY criterion based on the elastoplastic analysis is proposed in this study. Although the CTOD is one candidate as the parameter of the elastoplastic problem, the SIF is more practical as the parameter in general. The evaluation of the crack growth rate using the CTOD is fundamentally equivalent to using the SIF $K(a+\omega)$ at the crack length a in terms of the elastoplastic solution. Based on this rationale, the practical SSY criterion in the presence of the residual stress was established. The consideration is the following.

The SSY criterion in the case of $K(a+\omega)$ at the crack length a are the following, as inferred from Eq. (15):

$$K(a+\omega) \leqslant \sigma_Y\sqrt{0.4a}. \tag{19}$$

$K(a+\omega)$ can be rearranged as:

$$\begin{aligned} K(a+\omega) &= K_0(a+\omega) + K_{res}(a+\omega) \\ &\approx K_0(a) + \omega\frac{\partial K_0}{\partial a} + K_{res}(a) + \omega\frac{\partial K_{res}}{\partial a} \end{aligned}, \tag{20}$$

$K(a)$ can be written as:

$$K(a) = K_0(a) + K_{res}(a). \tag{21}$$

The SSY conditions are satisfied and the following equation is derived if the plastic zone size determined solely from the remote uniform stress is sufficiently small relative to the crack length:

$$K_0(a) \ll \omega\frac{\partial K_0}{\partial a}. \tag{22}$$

As an approximation formula, $K(a+\omega)$ can be transformed from Eqs. (20)–(22) as:

$$K(a+\omega) \approx K(a) + \omega\frac{\partial K_{res}}{\partial a}. \tag{23}$$

Therefore, the SSY criterion is derived as:

$$K(a) + \omega\frac{\partial K_{res}}{\partial a} \leqslant \sigma_Y\sqrt{0.4a}. \tag{24}$$

When the second term on the left, which is the SIF variation due to the residual stress at the plastic zone, is sufficiently small relative to $K(a)$, the SSY conditions are satisfied and the second term can be negligible. However, when the SIF variation is sufficiently large at the plastic zone, then the second term cannot be ignored. It has the potential to exceed the SSY criterion.

In the Dugdale model, the plastic zone size under the SSY conditions is written as follows for non-redistribution of the residual stress [10]:

$$\omega = \frac{\pi}{8}\left(\frac{K(a)}{\lambda\sigma_Y}\right)^2. \tag{25}$$

From Eqs. (24) and (25), the SSY criterion is pressed solely by the SIF in the case with a steep residual stress distribution:

$$K(a) + \frac{\pi}{8}\left(\frac{K(a)}{\lambda\sigma_Y}\right)^2\frac{\partial K_{res}}{\partial a} \leqslant \sigma_Y\sqrt{0.4a}. \tag{26}$$

The solid lines in Fig. 9 present the distribution parameters determined by the proposed SSY criterion of Eq. (26). The dashed lines represent analytical results portrayed in Fig. 4. The two results were consistent for $M = 0.2$. The error for the two became large for $M = 0.3$, and all the results of $M = 0.5$ exceeded the SSY criterion of Eq. (26). When M becomes larger, the width of the residual stress distribution becomes smaller than the plastic zone size. Consequently, the error from the second term on the left of Eq. (26) becomes large, which engenders the error in the two. However, when M became large, the results from Eq. (26) were evaluated conservatively as $M = 0.3$ shown in Fig. 9, which means that the SSY criterion from Eq. (26) is effective as a practical evaluation method.

In this study, the SSY conditions with steep residual stress distribution were discussed using the Dugdale model. However, the Irwin model might be applicable to similar analysis using the theory of plastic zone correction.

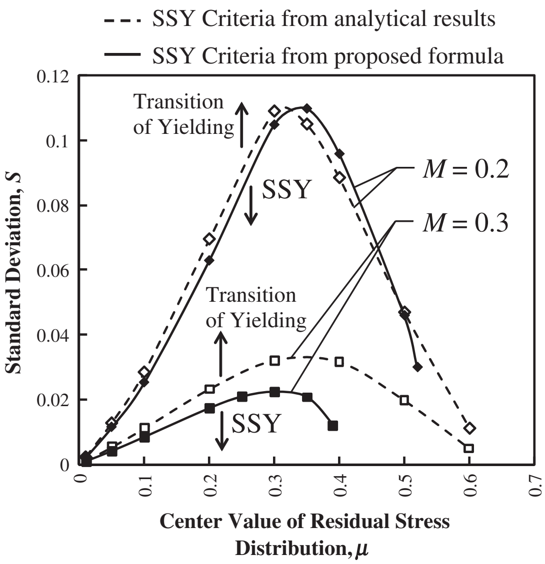

Fig. 9. SSY criteria according to Eq. (26), the formula used with residual stress. Dashed lines show the analytical results presented in Fig. 5.

4. Conclusions

The plastic zone size varies depending on the shape of the stress distribution along the oncoming crack propagation path under elastoplastic analysis. Consequently, a threshold value of the stress distribution parameters, which exceeds the SSY criterion, is expected to exist. In this study, a normal distribution was assumed as the residual stress distribution. Various shapes of the residual stress distributions along the crack propagation path were considered using Newman's two-dimensional model, which can be used for elastoplastic analysis. The following are the salient findings of this study.

The features of the residual stress distribution shape which influence the scale of yielding were clarified using numerical analysis. Results show that the SSY criterion tends to be exceeded when the gradient of the distribution becomes steep or the maximum value of the distribution becomes large.

When the steep residual stress distribution is present, the CTOD calculated using elastoplastic analysis increased before the crack tip reached the position where the distribution was present. In such a case, it is assumed to increase the crack growth rate of time-dependent fractures (fatigue or stress corrosion cracking) before the crack tip reaches such a position. Since elastic analysis is generally used for the evaluation of the remaining life of the structures, there is the potential to underestimate the remaining life when stress distribution exists along the oncoming crack propagation path.

The plastic zone size cannot be evaluated using elastic analysis when a steep residual stress distribution is present near the crack tip. Therefore, the traditional SSY criterion based on the elastic analysis is inadequate in the presence of a steep residual stress distribution. For this problem, a simplified practical SSY criterion based on the elastoplastic analysis, that is useful in the presence of the residual stress distribution, was proposed. The usefulness of the proposed method was demonstrated by comparing the results from the proposed method and Newman's analytical model.

Acknowledgement

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