内圧円筒の弾塑性解

1．両端閉じ内圧円筒

(1) 弾性解、ラーメの式

$$\sigma_r = p\frac{1}{\left(\frac{b}{a}\right)^2 - 1}\left[1 - \left(\frac{b}{r}\right)^2\right]$$

$$\sigma_\theta = p\frac{1}{\left(\frac{b}{a}\right)^2 - 1}\left[1 + \left(\frac{b}{r}\right)^2\right]$$

$$\tau_{r\theta} = 0$$

$$\sigma_z = p\frac{1}{\left(\frac{b}{a}\right)^2 - 1} = \frac{\sigma_r + \sigma_\theta}{2} \quad \text{（両端閉じ）}$$

(2) 弾塑性解

仮定

弾完全塑性体

弾塑性境界　$r = c$

$r > c$　弾性域

$r < c$　塑性域（ひずみの弾性成分無視）

弾性域だけ取り出すと，$r=c$: $\sigma_{rc} \rightarrow -p_c$, $\sigma_{\theta c}$, σ_{zc}

$r > c$（弾性域）

$$\sigma_r = p_c\frac{1}{\left(\frac{b}{c}\right)^2 - 1}\left[1 - \left(\frac{b}{r}\right)^2\right]$$

$$\sigma_\theta = p_c\frac{1}{\left(\frac{b}{c}\right)^2 - 1}\left[1 + \left(\frac{b}{r}\right)^2\right]$$

$$\sigma_z = p_c\frac{1}{\left(\frac{b}{c}\right)^2 - 1}$$

$r = c$

$$\sigma_{r=c} = -p_c$$

$$\sigma_{\theta c} = p_c\frac{1}{\left(\frac{b}{c}\right)^2 - 1}\left[1 + \left(\frac{b}{c}\right)^2\right]$$

$$\sigma_{zc} = p_c\frac{1}{\left(\frac{b}{c}\right)^2 - 1}$$

ミーゼスの降伏条件

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \left[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 \right]^{1/2}$$

$$= \sigma_Y$$

$\sigma_z = \dfrac{\sigma_r + \sigma_\theta}{2}$ を代入

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \left| \sigma_r - \sigma_\theta \right| = \sigma_Y$$

$$\frac{\sqrt{3}}{2} \left| \sigma_{rc} - \sigma_{\theta c} \right|$$

$$= \frac{\sqrt{3}}{2} \left| -p_c - p_c \frac{1}{\left(\frac{b}{c}\right)^2 - 1} \left[1 + \left(\frac{b}{c}\right)^2 \right] \right|$$

$$= \frac{\sqrt{3}}{2} \left[p_c \frac{2\left(\frac{b}{c}\right)^2}{\left(\frac{b}{c}\right)^2 - 1} \right]$$

$$= \sigma_Y$$

$$-\sigma_{rc} = p_c = \frac{\left(\frac{b}{c}\right)^2 - 1}{\sqrt{3}\left(\frac{b}{c}\right)^2} \sigma_Y = \frac{1 - \left(\frac{c}{b}\right)^2}{\sqrt{3}} \sigma_Y$$

(3) 降伏開始圧 p_i

$c \rightarrow a$

$$p_i = \frac{1 - \left(\frac{a}{b}\right)^2}{\sqrt{3}} \sigma_Y$$

(4) 応力

塑性でも $\sigma_z = \dfrac{\sigma_r + \sigma_\theta}{2}$ を仮定

弾性 $\varepsilon_z = 0 \rightarrow \sigma_z = \nu(\sigma_r + \sigma_\theta)$

塑性 $d\varepsilon_z = 0,\ \nu = \dfrac{1}{2} \rightarrow \sigma_z = \dfrac{1}{2}(\sigma_r + \sigma_\theta)$

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \left| \sigma_r - \sigma_\theta \right| = \sigma_Y$$

or

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \left| \sigma_\theta - \sigma_r \right| = \sigma_Y$$

ミーゼス $2\tau_Y = \dfrac{2}{\sqrt{3}} \sigma_Y$

$\sigma_\theta > 0,\ \sigma_r < 0$

$$\sigma_\theta - \sigma_r = 2\tau_Y = \frac{2}{\sqrt{3}}\sigma_Y$$

r < c（塑性域）

つり合い方程式

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{d\sigma_r}{dr} - \frac{2}{\sqrt{3}}\frac{\sigma_Y}{r} = 0$$

解

$$\sigma_r = \frac{2}{\sqrt{3}}\sigma_Y \ln r + C \quad (C：定数)$$

境界条件

(1) $r = a \rightarrow \sigma_r = -p$

(2) $r = c \rightarrow \sigma_{rc}$（弾性域）$= \sigma_{rc}$（塑性域）

(3) $r = b \rightarrow \sigma_r = 0$

(4) $r = c$の弾性域の応力は降伏条件を満足

$r = c$

$$\sigma_{rc} = \frac{2}{\sqrt{3}}\sigma_Y \ln c + C \qquad (塑性域)$$

$$\sigma_{rc} = -p_c = -\frac{1-\left(\frac{c}{b}\right)^2}{\sqrt{3}}\sigma_Y \qquad (弾性域)$$

$$\frac{2}{\sqrt{3}}\sigma_Y \ln c + C = -\frac{1-\left(\frac{c}{b}\right)^2}{\sqrt{3}}\sigma_Y$$

$$C = -\frac{\sigma_Y}{\sqrt{3}}\left[2\ln c + 1 - \left(\frac{c}{b}\right)^2\right]$$

応力

$$\sigma_r = \frac{\sigma_Y}{\sqrt{3}}\left[2\ln\frac{r}{c} - 1 + \left(\frac{c}{b}\right)^2\right]$$

$$\sigma_\theta = \sigma_r + \frac{2}{\sqrt{3}}\sigma_Y$$

$$= \frac{\sigma_Y}{\sqrt{3}}\left[2\ln\frac{r}{c} + 1 + \left(\frac{c}{b}\right)^2\right]$$

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2}$$

$$= \frac{\sigma_Y}{\sqrt{3}}\left[2\ln\frac{r}{c} + \left(\frac{c}{b}\right)^2\right]$$

(5) 圧力と弾塑性境界

$r \rightarrow a,\ \sigma_r = -p$

$$p = -\frac{\sigma_Y}{\sqrt{3}}\left[2\ \ln\frac{a}{c} - 1 + \left(\frac{c}{b}\right)^2\right]$$

$$= \frac{2}{\sqrt{3}}\sigma_Y\left[\ \ln\frac{c}{a} + \frac{1}{2}\left\{1 - \left(\frac{c}{b}\right)^2\right\}\right]$$

(6) 軸方向の力のつり合い

軸方向の力 F

$$F = \int_a^b \sigma_z\, 2\pi r dr = \int_a^b \pi r\ (\ \sigma_r + \sigma_\theta\ ) dr$$

$$\because\ \sigma_z = \frac{\sigma_r + \sigma_\theta}{2}$$

つり合い方程式

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\sigma_\theta\ dr = r d\sigma_r + \sigma r dr = d(r\sigma_r)$$

$$\therefore\ F = \int_a^b \pi r\ (\ \sigma_r dr + \sigma_\theta\ dr\ )$$

$$= \int_a^b \pi r\ [\ (\sigma_r dr + d(rdr)\ ]$$

$$= \int_a^b \pi\, d(r^2\sigma_r)$$

$$= \pi r^2\ \sigma_r \Big|_a^b$$

~~$= \pi a^2 p$~~ $= \pi(b^2 - a^2) P$　　満足

(7) 塑性崩壊圧 p_f

$c \rightarrow b$, 全断面降伏

$$p_Y = \frac{2}{\sqrt{3}}\sigma_Y\ \ln\frac{b}{a}$$

降伏開始

$$p_i = \frac{1}{\sqrt{3}}\sigma_Y\left[\ 1 - \left(\frac{a}{b}\right)^2\ \right]$$

塑性崩壊（弾完全塑性体）

$$p_f = \frac{2}{\sqrt{3}}\sigma_Y\ \ln\frac{b}{a}$$

塑性崩壊＋破壊（ひずみ硬化塑性体：降伏応力 σ_Y, 引張強さ σ_B）

$$p_Y = \frac{2}{\sqrt{3}}\sigma_Y\ \ln\frac{b}{a}$$

$$p_B = \frac{2}{\sqrt{3}} \sigma_B \ln \frac{b}{a}$$

$$p_f = p_Y + \frac{\sigma_Y}{\sigma_B} (p_B - p_Y)$$

$$= \frac{\sigma_Y}{\sigma_B} p_B + (1 - \frac{\sigma_Y}{\sigma_B}) p_Y$$

$$= p_Y + (1 - \frac{\sigma_Y}{\sigma_B}) p_Y$$

$$= (2 - \frac{\sigma_Y}{\sigma_B}) p_Y$$

$$= \frac{2}{\sqrt{3}} (2 - \frac{\sigma_Y}{\sigma_B}) \sigma_Y \ln \frac{b}{a}$$ Faupelの式

２．両端開き内圧円筒

$\sigma_z = 0$

ミーゼスの降伏条件

$$\bar{\sigma} = \frac{1}{\sqrt{2}} [(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2]^{1/2}$$

$$= \sigma_Y$$

$$\bar{\sigma} = \sqrt{\sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2} = \sigma_Y$$

$$\sigma_r = \frac{1 - (\frac{b}{r})^2}{(\frac{b}{a})^2 - 1} p$$ 弾性解，ラーメの式

$$\sigma_\theta = \frac{1 + (\frac{b}{r})^2}{(\frac{b}{a})^2 - 1} p$$

$$\therefore p \frac{\sqrt{3 (\frac{a}{r})^4 + (\frac{a}{b})^4}}{1 - (\frac{a}{b})^2} = \sigma_Y$$

$r = a$，降伏開始圧

$$p_I = \frac{1 - (\frac{a}{b})^2}{\sqrt{3 + (\frac{a}{b})^4}} \sigma_Y$$

3．両端閉じ内圧円筒（トレスカの降伏条件）

$$\bar{\sigma} = \sigma_\theta - \sigma_r = \sigma_Y$$

cf．ミーゼス

$$\bar{\sigma} = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r) = \sigma_Y$$

ミーゼスの解 → トレスカの解

$$\frac{2}{\sqrt{3}}\sigma_Y \quad \rightarrow \quad \sigma_Y$$

降伏開始圧

$$p_i = \frac{1-\left(\frac{a}{b}\right)^2}{\sqrt{3}}\sigma_Y \rightarrow p_i = \frac{1-\left(\frac{a}{b}\right)^2}{2}\sigma_Y$$

$$\sigma_\theta = \frac{\sigma_Y}{\sqrt{3}}\left[2\ln\frac{r}{c} + 1 + \left(\frac{c}{b}\right)^2\right]$$

$$\rightarrow \quad \sigma_\theta = \frac{\sigma_Y}{2}\left[2\ln\frac{r}{c} + 1 + \left(\frac{c}{b}\right)^2\right]$$

$$p_f = \frac{2\sigma_Y}{\sqrt{3}}\ln\frac{b}{a} \rightarrow p_f = \sigma_Y \ln\frac{b}{a}$$