軸対称問題

１．円柱極座標（図１参照）

$$r^2 = x^2 + y^2$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right) \quad \tan\theta = \frac{y}{x}$$

z

２．ひずみ（図２参照）

A → A'の変位

r方向　u_r

θ方向　u_θ

(1) r方向ひずみ　ε_r

A'の変位u_r、B'の変位$u_r + \dfrac{\partial u_r}{\partial r} dr$

$$\varepsilon_r = \frac{\left(u_r + \dfrac{\partial u_r}{\partial r} dr\right) - u_r}{dr} = \frac{\partial u_r}{\partial r}$$

(2) θ方向ひずみε_θ

① r方向変位によるθ方向ひずみ（図３参照）

$$\frac{(r+u_r)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u_r}{r}$$

② θ方向変位によるθ方向ひずみ（図４参照）

A'の変位u_θ、D'の変位$u_\theta + \dfrac{\partial u_\theta}{\partial \theta} d\theta$

$$\frac{\left(u_\theta + \dfrac{\partial u_\theta}{\partial \theta} d\theta\right) - u_\theta}{r\,d\theta} = \frac{1}{r}\ \frac{\partial u_\theta}{\partial \theta}$$

$$\therefore \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\ \frac{\partial u_\theta}{\partial \theta}$$

(3) せん断ひずみ$\gamma_{r\theta}$

A → A'のr方向変位　u_r

D → D'のr方向変位　$u_r + \dfrac{\partial u_r}{\partial \theta} d\theta$

A'D'間のr方向変位差　$\dfrac{\partial u_r}{\partial \theta} d\theta$

AD、 A'D'の角度

$$\frac{\frac{\partial u_r}{\partial \theta} d\theta}{r d\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta}$$

A → A'のθ方向変位　u_θ

B → B'のθ方向変位　$u_\theta + \frac{\partial u_\theta}{\partial r} dr$

A'B'間のθ方向変位差　$\frac{\partial u_\theta}{\partial r} dr$

AB、 A'B'の角度

$$\frac{\frac{\partial u_\theta}{\partial r} dr}{dr} = \frac{\partial u_\theta}{\partial r}$$

ただし、 この中には要素全体の回転角が含まれている。

$$\frac{u_\theta}{r}$$

$$\therefore \gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta}$$

3. 応力（図５参照）

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r)$$

$$\gamma_{r\theta} = \frac{1}{G} \tau_{r\theta}$$

4. 応力のつり合い方程式

(1) r方向応力 σ_r（r方向、半径方向、径方向）（図６参照）

r+drとrでの面積差、 dz： 単位厚さ

$$(r+dr)d\theta dz - r d\theta dz = dr d\theta dz$$

面積差に起因する力の差

$$\sigma_r dr d\theta dz$$

応力勾配に起因する力の差

$$\frac{\partial \sigma_r}{\partial r} dr \ (r d\theta) \ dz$$

r方向の力の差

$$\left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} \right) r dr d\theta dz$$

(2) θ方向応力 σ_θ（θ方向、接線方向、（円周方向、周方向））（図7参照）

r方向成分の力

$$\left(\frac{1}{2}\sigma_\theta d\theta + \frac{1}{2}\sigma_\theta d\theta\right) drdz = \sigma_\theta drd\theta dz$$

r方向に直角な成分の力

面積差なし、応力勾配に起因する力の差

$$\frac{\partial \sigma_\theta}{\partial \theta} d\theta (drdz)$$

(3) せん断応力 $\tau_{r\theta}$（図8参照）

r方向成分の力

上下で面積差なし

θ方向の応力勾配に起因する力

$$\frac{\partial \tau_{r\theta}}{\partial \theta} d\theta (drdz)$$

r方向に直角な成分の力（上下）

$$\left(\frac{1}{2}\tau_{r\theta} d\theta + \frac{1}{2}\tau_{r\theta} d\theta\right) drdz = \tau_{r\theta} d\theta drdz$$

r方向に直角な成分の力（左右）

r+drとrでの面積差　$drd\theta dz$

面積差に起因する力の差　$\tau_{r\theta} drd\theta dz$

応力勾配に起因する力の差　$\dfrac{\partial \tau_{r\theta}}{\partial r} dr\,(rd\theta)\,dz$

力の差　$\left(\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{\tau_{r\theta}}{r}\right) rdrd\theta dz$

(4) r方向の力のつり合い

$$\left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r}\right) rdrd\theta dz - \sigma_\theta drd\theta dz + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta drdz = 0$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

(5) r方向に直角な力のつり合い

$$\frac{\partial \sigma_\theta}{\partial \theta} d\theta drdz + \tau_{r\theta} d\theta drdz + \left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\tau_{r\theta}}{r}\right) rdrd\theta dz = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0$$

5．Airyの応力関数

(1) 座標変換（図９参照）

$$(\sigma_x,\ \sigma_y,\ \tau_{xy}) \rightarrow (\sigma_r,\ \sigma_\theta,\ \tau_{r\theta})$$

$$\begin{cases} \sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \\ \sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy} \sin 2\theta \\ \tau_{r\theta} = \dfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

(2) Airyの応力関数 ϕ

$$\begin{cases} \sigma_x = \dfrac{\partial^2\phi}{\partial y^2} \\ \sigma_y = \dfrac{\partial^2\phi}{\partial x^2} \\ \tau_{xy} = -\dfrac{\partial^2\phi}{\partial x \partial y} \end{cases}$$

$$\begin{cases} \sigma_r = \left(\cos^2\theta \dfrac{\partial^2}{\partial y^2} + \sin^2\theta \dfrac{\partial^2}{\partial x^2} - \sin 2\theta \dfrac{\partial^2}{\partial x \partial y}\right)\phi \\ \sigma_\theta = \left(\sin^2\theta \dfrac{\partial^2}{\partial y^2} + \cos^2\theta \dfrac{\partial^2}{\partial x^2} + \sin 2\theta \dfrac{\partial^2}{\partial x \partial y}\right)\phi \\ \tau_{r\theta} = \left\{\dfrac{1}{2}\sin 2\theta \left(\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}\right) - \cos 2\theta \dfrac{\partial^2}{\partial x \partial y}\right\}\phi \end{cases}$$

$$r^2 = x^2 + y^2 \ \underset{\text{xで偏微分}}{\rightarrow}\ 2r\frac{\partial r}{\partial x} = 2x$$

$$\underset{\text{yで偏微分}}{\rightarrow}\ 2r\frac{\partial r}{\partial y} = 2y$$

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta \\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \end{cases}$$

$$\tan\theta = \frac{y}{x} \ \underset{\text{xで偏微分}}{\rightarrow}\ \sec^2\theta \frac{\partial\theta}{\partial x} = -\frac{y}{x^2}$$

$$\underset{\text{yで偏微分}}{\rightarrow}\ \sec^2\theta \frac{\partial\theta}{\partial y} = \frac{1}{x}$$

$$\begin{cases} \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{x^2}\cos^2\theta = -\dfrac{\sin\theta}{r} \\ \dfrac{\partial \theta}{\partial y} = \dfrac{\cos^2\theta}{x} = \dfrac{\cos\theta}{r} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} = \cos\theta\dfrac{\partial}{\partial r}\left(\dfrac{\partial}{\partial x}\right) - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial}{\partial x}\right) \\ \qquad = \cos^2\theta\dfrac{\partial^2}{\partial r^2} - \dfrac{\sin 2\theta}{r}\dfrac{\partial^2}{\partial r\partial\theta} + \dfrac{\sin^2\theta}{r}\dfrac{\partial}{\partial r} \\ \qquad + \dfrac{\sin 2\theta}{r^2}\dfrac{\partial}{\partial\theta} + \dfrac{\sin^2\theta}{r^2}\dfrac{\partial^2}{\partial\theta^2} \\ \dfrac{\partial^2}{\partial y^2} = \sin\theta\dfrac{\partial}{\partial r}\left(\dfrac{\partial}{\partial y}\right) + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial}{\partial y}\right) \\ \qquad = \sin^2\theta\dfrac{\partial^2}{\partial r^2} + \dfrac{\sin 2\theta}{r}\dfrac{\partial^2}{\partial r\partial\theta} + \dfrac{\cos^2\theta}{r}\dfrac{\partial}{\partial r} \\ \qquad - \dfrac{\sin 2\theta}{r^2}\dfrac{\partial}{\partial\theta} + \dfrac{\cos^2\theta}{r^2}\dfrac{\partial^2}{\partial\theta^2} \\ \dfrac{\partial^2}{\partial x\partial y} = \cos\theta\dfrac{\partial}{\partial r}\left(\dfrac{\partial}{\partial y}\right) - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial}{\partial y}\right) \\ \qquad = \dfrac{1}{2}\sin 2\theta\dfrac{\partial^2}{\partial r^2} - \dfrac{1}{2}\dfrac{\sin 2\theta}{r^2}\dfrac{\partial^2}{\partial\theta^2} \\ \qquad + \dfrac{\cos 2\theta}{r}\dfrac{\partial^2}{\partial r\partial\theta} - \dfrac{\cos 2\theta}{r^2}\dfrac{\partial}{\partial\theta} - \dfrac{\sin 2\theta}{2r}\dfrac{\partial}{\partial r} \end{cases}$$

代入

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} \\ \sigma_\theta = \dfrac{\partial^2\phi}{\partial r^2} \\ \tau_{r\theta} = \dfrac{1}{r^2}\dfrac{\partial\phi}{\partial\theta} - \dfrac{1}{r}\dfrac{\partial^2\phi}{\partial r\partial\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right) \end{cases}$$

(3) つり合い方程式を満足

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0$$

(4) Laplace演算子

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(5) 重調和方程式

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

$$\times \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$

$$= 0$$

cf.

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0$$

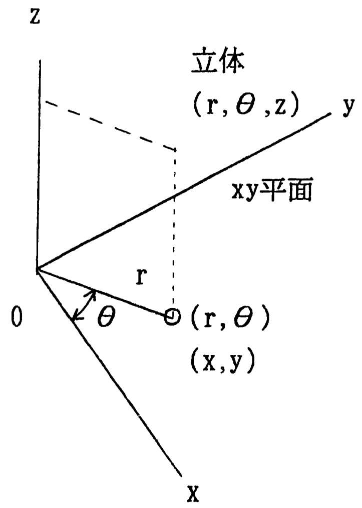
z
立体
(r, θ ,z)
y
xy平面
r
0
θ
(r, θ)
(x,y)
x

図1

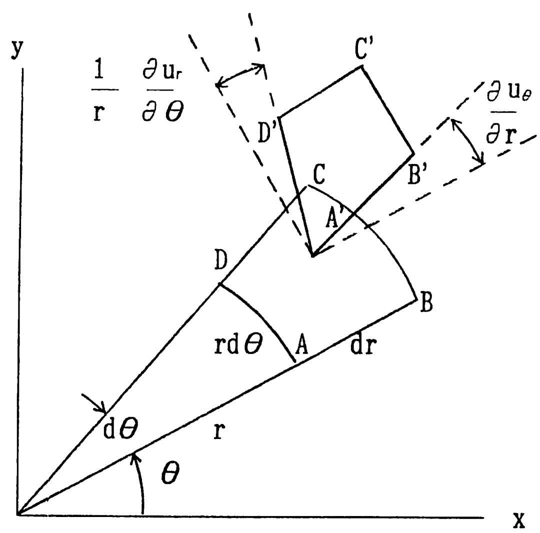
y
1/r ∂ur/∂θ
C'
∂uθ/∂r
D'
C
B'
A'
D
B
rdθ
A
dr
dθ
r
θ
x

図2

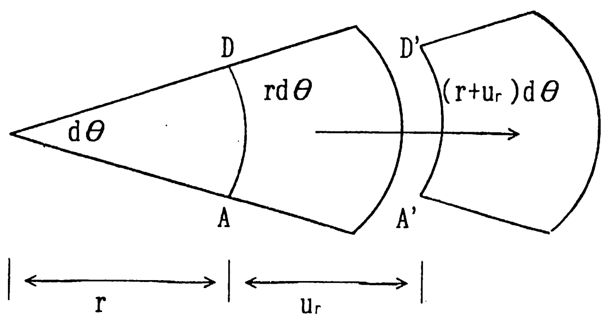
D
D'
rdθ
(r+ur)dθ
dθ
A
A'
r
ur

図3

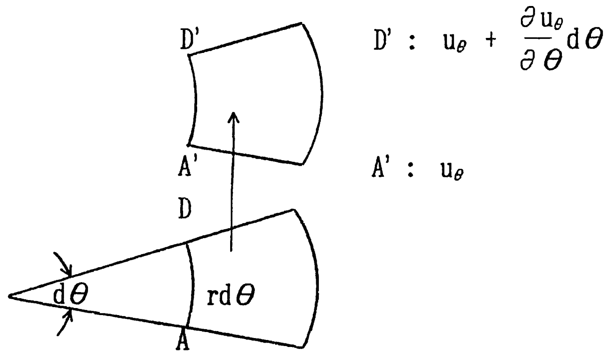
D'
A'
D
A
dθ
rdθ
D' : $u_\theta + \frac{\partial u_\theta}{\partial \theta} d\theta$
A' : u_θ

図4

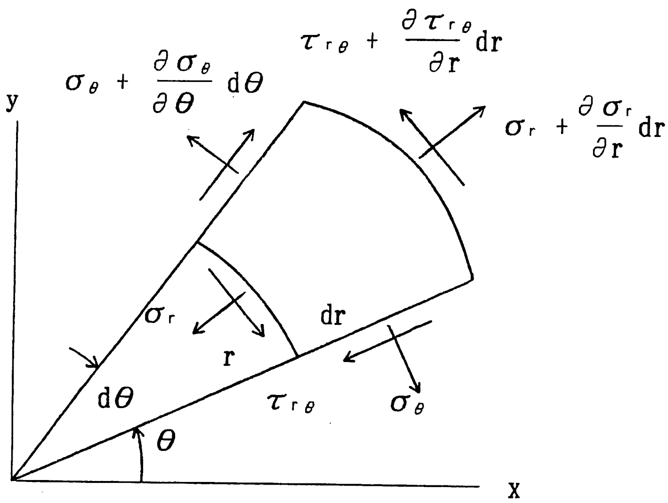
$\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr$
$\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta$
y
$\sigma_r + \frac{\partial \sigma_r}{\partial r} dr$
σ_r
dr
r
dθ
$\tau_{r\theta}$
σ_θ
θ
x

図5

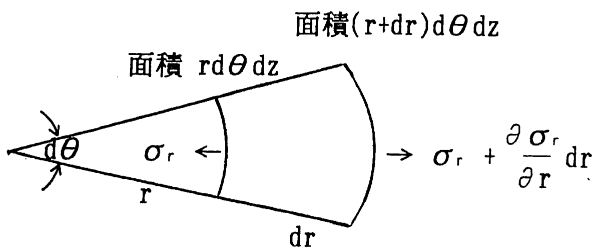
面積(r+dr)dθdz
面積 rdθdz
dθ
σ_r ←
→ $\sigma_r + \frac{\partial \sigma_r}{\partial r} dr$
r
dr

図6

$$(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta)\sin\frac{d\theta}{2}$$

$$\fallingdotseq \sigma_\theta \frac{d\theta}{2} + \frac{\partial \sigma_\theta}{\partial \theta} d\theta \frac{d\theta}{2}$$

$$\fallingdotseq \frac{1}{2}\sigma_\theta d\theta$$

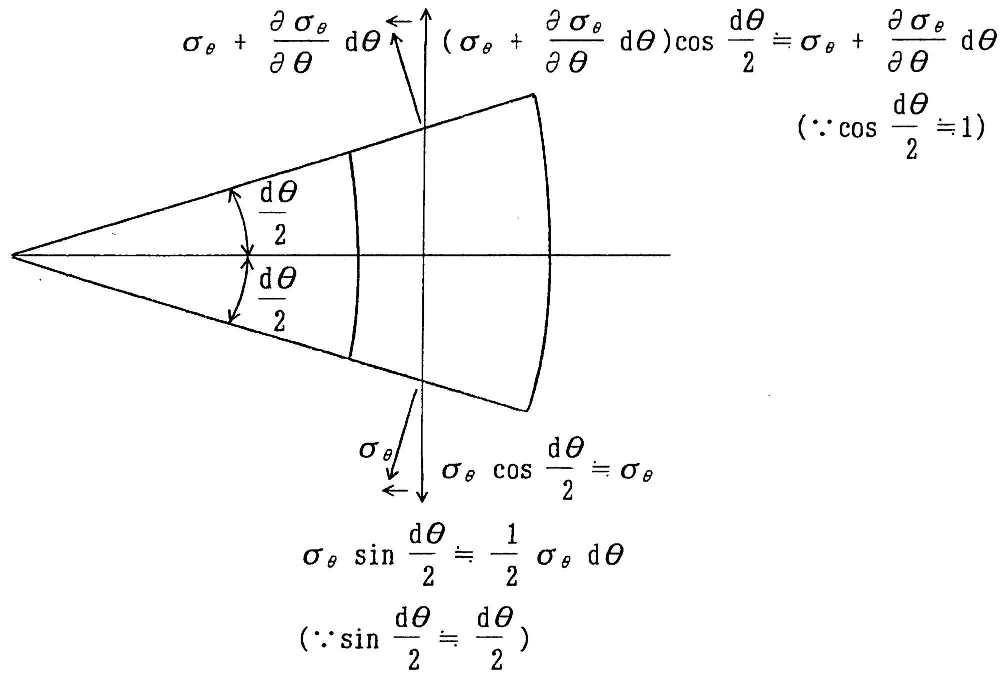

図7

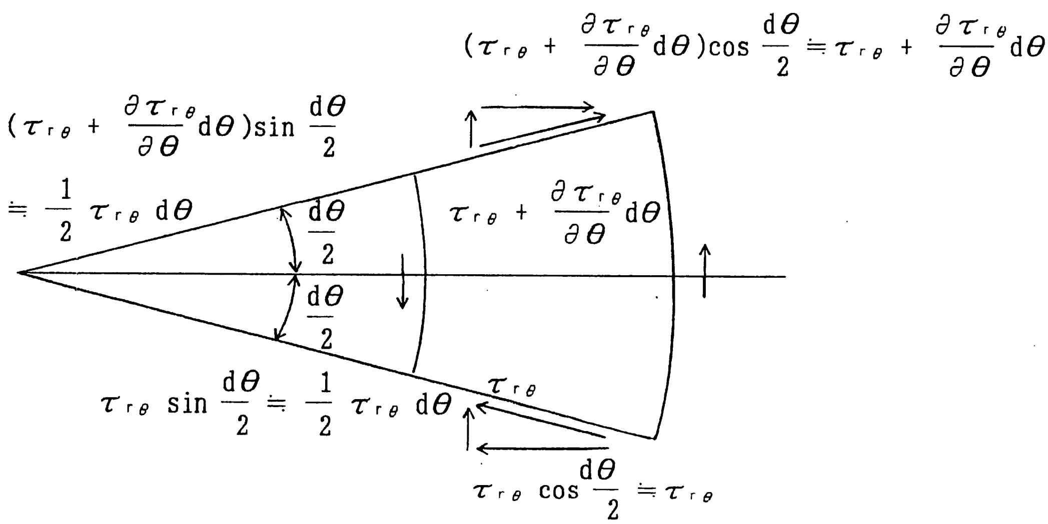

図8

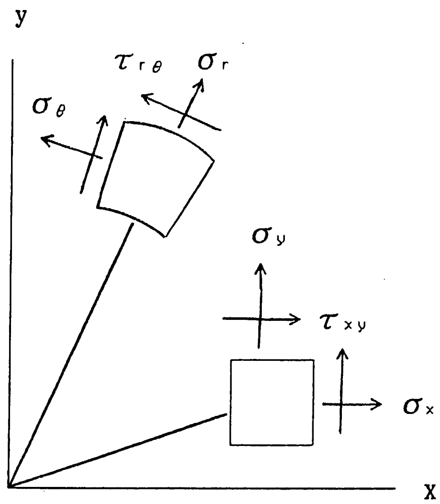

図9